

**SOME FRACTIONAL EXTENSIONS OF
THE TEMPERATURE FIELD PROBLEM IN OIL STRATA**

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*Dedicated to Prof. Michele Caputo
on the occasion of his 80th birthday*

Abstract

This survey is devoted to some fractional extensions of the incomplete lumped formulation, the lumped formulation and the formulation of Lauwrier of the temperature field problem in oil strata. The method of integral transforms is used to solve the corresponding boundary value problems for the fractional heat equation. By using Caputo's differintegration operator and the Laplace transform, new integral forms of the solutions are obtained. In each of the different cases the integrands are expressed in terms of a convolution of two special functions of Wright's type.

Mathematics Subject Classification: Caputo differintegration operator; fractional heat equation; fractional integrals and derivatives; Laplace transforms; Wright's function

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1. Introduction

A porous medium (sandstone) saturated with oil is called an oil stratum. It is possible to consider the stratum depth as equal to infinity since the depth of oil stratum usually varies from one to several kilometers. The rock surrounding a stratum (cap and base rock) is considered impermeable to the fluid. A standard method of oil extraction is to pump the oil out from a series of production wells which are drilled in the center of the oil deposit. At particular time of the exploitation period hot fluid is injected into wells drilled along the boundary of the oil reservoir.

The problem of describing the temperature field $u = u(x, y, z, t)$ of an oil strata arises when the hot fluid is injected in order to facilitate the oil extraction process. The heat equation for a porous medium is derived under the following general assumptions on the model ([1]): the temperature of the porous medium is equal to the temperature of the fluid that fills the strata; the thermal properties of both the cap and the base rock are identical; the boundary temperatures and the heat fluxes are equal on each side of the interface between the strata and its surrounding media; and the strata is horizontal and has constant depth. According to the assumptions the consideration of the temperature field is restricted to the cap rock only, and by choosing $z = 0$ the temperature on the surface of the strata is obtained.

Two cases of fluid injections, linear and radial, are mainly considered. In the *linear case*, a hot fluid is forced into the strata in the positive and negative x -directions with constant velocity through an infinitely long vertical gallery. The strata is supposed to be situated in the infinite slab $-\infty < x, y < \infty$, $-H \leq z < 0$ ($H > 0$), the cap rock is in upper half space $-\infty < x, y < \infty$, $0 < z < \infty$, and the base rock is in the lower half space $-\infty < x, y < \infty$, $-\infty < z \leq -H$.

In the *radial case*, where polar coordinates r and φ are used instead of x and y , a hot fluid is forced through an infinitely thin well, which is considered as a linear source of incompressible fluid with positive volume rate. The temperatures are considered along the axis of the injection wells or in the plane of the injection gallery.

Beside the exact formulation of the problem, the following three approximate formulations are treated (Antimirov et al. [1]): the *lumped formulation*, where the terminal conductivity of the strata is infinitely large in the vertical direction; the *incomplete lumped formulation*, where horizontal

heat transfer in the cap and base rock is neglected; and the *formulation of Lauwerier*, where horizontal heat transfer in the strata is also neglected.

In the radial case of the incomplete lumped formulation, the temperature field $u = u(r, z, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial z^2}, \quad 0 < r, z, t < \infty, \quad a > 0, \quad (1)$$

subject to the nonstandard boundary condition

$$z = 0 : \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1 - 2\nu}{r} \frac{\partial u}{\partial r} + \alpha \frac{\partial u}{\partial z}, \quad 0 < r, t < \infty \quad (2)$$

and the additional conditions

$$\begin{aligned} & \text{(a) } z = 0 \text{ and } r = 0 : u = 1 \\ & \text{(b) if } r^2 + z^2 \rightarrow \infty, \text{ then } u \rightarrow 0, \text{ and} \\ & \text{(c) } t = 0 : u = 0. \end{aligned} \quad (3)$$

The constant $a > 0$ depends on the coefficient of thermal diffusivity of the cap rock and the strata. The constant $\alpha > 0$ is a ratio of the coefficients of thermal conductivity of the cap rock and the strata. The constant $\nu > 0$ depends on the volume rate and the volumetric heat capacity of the fluid as well as the coefficient of the thermal conductivity of the strata.

In the linear case of the incomplete lumped formulation, the temperature field $u = u(x, z, t)$ in cartesian coordinates (vertical xz -plane) satisfies the heat equation (1) subject to the nonstandard boundary condition

$$z = 0 : \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2\gamma \frac{\partial u}{\partial x} + \alpha \frac{\partial u}{\partial z}, \quad 0 < x, t < \infty \quad (4)$$

and the additional conditions

$$\begin{aligned} & \text{(a) } z = 0 \text{ and } x = 0 : u = 1 \\ & \text{(b) if } x^2 + z^2 \rightarrow \infty, \text{ then } u \rightarrow 0, \text{ and} \\ & \text{(c) } t = 0 : u = 0. \end{aligned} \quad (5)$$

The constant $\gamma > 0$ depends on the volume rate of fluid per unit gallery length, the coefficient of the volumetric specific heat of the fluid and the coefficient of thermal conductivity of the strata.

In the lumped formulation, for sufficiently large injection velocity ($\gamma \rightarrow \infty$), the condition (5a) is replaced by

$$z = 0 \text{ and } x = 0 : \frac{\partial u}{\partial x} - 2\gamma u = -2\gamma. \quad (6)$$

The Lauwerier formulation relates to the temperature field of a single layer stratum in the case the velocity of the heat transfer between the fluid and the skeleton is finite. In this case, one has to consider separately the temperature $u(x, t)$ and $\Theta(x, z, t)$ of the fluid and the cap rock, respectively. The heat equation for the region containing the fluid and the skeleton respectively are derived [1, 8.7] under the main assumption that instead of having two regions containing the fluid and the skeleton separately, there is just a single region which is taken to be a porous medium. For a sufficiently large filtration velocity, one can neglect the heat transferred to the cap rock and stratum in the x -direction in comparison with the heat transfer in the z -direction. In this case the Lauwerier formulation for the linear fluid injection reads as

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial z^2}, \quad 0 < x, z, t < \infty \quad (7)$$

$$z = 0 : \frac{\partial u}{\partial t} = -\gamma \frac{\partial u}{\partial x} - \alpha(u - \Theta), \quad 0 < x, t < \infty \quad (8)$$

$$z = 0 : \frac{\partial \Theta}{\partial t} = \mu \frac{\partial \Theta}{\partial z} + k(u - \Theta), \quad 0 < x, t < \infty \quad (9)$$

$$\begin{aligned} \text{(a)} \quad & x = 0 : u = 1, \\ \text{(b)} \quad & u, \Theta \rightarrow 0 \quad \text{as} \quad x^2 + z^2 \rightarrow \infty, \\ \text{(c)} \quad & t = 0 : u = \Theta = 0, \end{aligned} \quad (10)$$

where the constant $\mu > 0$ depends on the coefficient of thermal conductivity of the cap rock, the volumic heat capacity of skeleton and the coefficient of thermal conductivity of the cap rock; the constant $k > 0$ depends on the porosity and the volumic heat capacity of the fluid and the skeleton.

By using the Laplace transform

$$L[f(t)] = \int_0^\infty e^{-pt} f(t) dt, \quad (11)$$

the solution of the problem (1), (2), (3) is given in the form ([1], 8.2.54)

$$u(r, z, t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{1}{\tau} \left(\frac{r^2}{4\tau} \right)^\nu e^{-\frac{r^2}{4\tau}} \operatorname{erfc} \left(\frac{\frac{z}{a} + b\tau}{2\sqrt{t-\tau}} \right) d\tau, \quad (12)$$

where $\Gamma(\nu)$ is Euler's gamma function, $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$ is the complementary error function and $b = \alpha/a$. Formula (12) is used in numerous

computations of oil strata temperature fields in the radial case, [1]. The Laplace transform method applied to the problem (1),(4) and (5) yields the solution

$$u(x, z, t) = \int_0^t \frac{x}{2\tau\sqrt{\pi\tau}} e^{-(x/2\sqrt{\tau}-\gamma\sqrt{\tau})^2} \operatorname{erfc}\left(\frac{\frac{z}{a} + b\tau}{2\sqrt{t-\tau}}\right) d\tau. \quad (13)$$

The temperature $u_s(x, t)$ of the strata is determined by (13) if $z = 0$. The same approach applied to the lumped formulation (1),(4),(6), and (5b) and (5c) results to

$$u(x, z, t) = 2\gamma e^{\gamma x} \int_0^t \left[\frac{1}{\sqrt{\pi t}} e^{-x^2/4\tau-\gamma^2\tau} - \gamma e^{\gamma x} \operatorname{erfc}\left(\frac{x}{2\sqrt{\tau}} + \gamma\sqrt{\tau}\right) \right] \operatorname{erfc}\left(\frac{\frac{z}{a} + b\tau}{2\sqrt{t-\tau}}\right) d\tau \quad (14)$$

and to $u_s(x, t)$ setting $z = 0$. Similar boundary value problems are also studied by using the Laplace and the general Hankel transforms [3].

By using the Laplace transform (11), the solution of the problem (7), (8), (9) and (10) is given by the formulas ([1], (8.7.40), (8.7.44))

$$u(x, t) = e^{-\frac{\alpha x}{\gamma}} \int_0^{t-\frac{x}{\gamma}} e^{-k\tau} \left[\frac{\mu(2t - \frac{2x}{\gamma} - \tau)}{2\sqrt{\pi}(t - \frac{x}{\gamma} - \tau)^{3/2}} e^{-\left(\frac{\tau\mu}{2\sqrt{t-\frac{x}{\gamma}-\tau}}\right)^2} + k \operatorname{erfc}\left(\frac{\tau\mu}{2\sqrt{t-\frac{x}{\gamma}-\tau}}\right) \right] I_0\left(2\sqrt{\frac{k\alpha x}{\gamma}\tau}\right) d\tau \quad (15)$$

and

$$\Theta(x, z, t) = k e^{-\frac{\alpha x}{\gamma}} \int_0^{t-\frac{x}{\gamma}} e^{-k\tau} \operatorname{erfc}\left(\frac{z + \mu\tau}{2\sqrt{t-\frac{x}{\gamma}-\tau}}\right) I_0\left(2\sqrt{\frac{k\alpha x}{\gamma}\tau}\right) d\tau, \quad (16)$$

where $I_0(z)$ is the modified Bessel function of the first kind with zero index.

The main goal of this paper is to formulate in a reasonable way and to solve the fractional generalizations of the above problems. Further on in the text we refer to these generalizations as fractional incomplete lumped formulation (the radial and linear case), fractional lumped formulation and fractional Lauwerier formulation of the temperature field problem in oil strata respectively.

2. Fractional heat equation

Fractional calculus is a significant topic in mathematical analysis as a result of its increasing range of applications. Operators for fractional differentiation and integration (differintegration operators) have been used in various fields such as: hydraulics of dams, potential fields, diffusion problems and waves in liquids and gases, [27]. The use of half-order derivatives and integrals leads to a formulation of certain electro-chemical problems which is more economical and useful than the classical approach in terms of Fick's law of diffusion [8]. The main advantage of the fractional calculus is that the fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. In special treatises like [24], [26], [20] and [25] the mathematical aspects and applications of the fractional calculus are extensively discussed.

For our purposes we adopt in this paper the Caputo fractional derivative [6],

$$D_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \\ \frac{d^m f(t)}{dt^m}, & \alpha = m, \end{cases} \quad (17)$$

where m is positive integer. This definition was introduced by Caputo in the late sixties of the twentieth century and adopted by Caputo and Mainardi in the framework of the theory of linear viscoelasticity. The method we follow makes the rule of the Laplace transform (11) very important (see [25])

$$L[D_*^\alpha f(t)] = p^\alpha L[f(t)] - \sum_{k=0}^{m-1} f^{(k)}(0) p^{\alpha-1-k}, \quad m-1 < \alpha \leq m. \quad (18)$$

The modeling of diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives [25], [18]. Two types of partial differential equations of fractional order deserve special attention. The first type is a generalization of the fractional partial differential equation suggested by Oldham and Spanier as a replacement of Fick's law [23]. The fractional-order diffusion equation suggested by Metzler, Glöckle and Nonnenmacher [19] is an example of the second type of fractional diffusion equation. Another example of the second type is the fractional diffusion equation deduced by Nigmatullin [21], [22] in the form

$$\frac{\partial^2 \beta u}{\partial t^{2\beta}} = a^2 \frac{\partial^2 u}{\partial z^2}, \quad 0 < \beta \leq \frac{1}{2} \quad (a > 0). \quad (19)$$

The equation (19) is also known as the *fractional diffusion-wave equation* (cf. [15], [16]). When the order of the fractional derivative is $2\beta = 1$, the equation becomes the classical diffusion equation, and if $2\beta = 2$ it becomes the classical wave equation. For $0 < 2\beta < 1$ we have the so-called ultraslow diffusion. Values $1 < 2\beta < 2$ correspond to the intermediate processes [13].

In this paper we consider the fractional diffusion equation (19) in the case when the fractional time derivative of the field temperature is given by the Caputo fractional derivative (17). Thus we extend the problem (1), (2), (3) to the following *radial case of the fractional incomplete formulation* that reads as

$$D_*^{2\beta} u = a^2 \frac{\partial^2 u}{\partial z^2}, \quad 0 < r, z, t < \infty, \quad 0 < \beta \leq \frac{1}{2}, \quad a > 0, \quad (20)$$

subject to the boundary condition

$$z = 0 : D_*^{2\beta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1 - 2\nu}{r} \frac{\partial u}{\partial r} + \alpha \frac{\partial u}{\partial z}, \quad 0 < r, t < \infty, \quad (21)$$

and the conditions

$$\begin{aligned} (a) \quad & r = 0, z = 0 : u = 1 \\ (b) \quad & \text{if } r^2 + z^2 \rightarrow \infty, \text{ then } u \rightarrow 0, \\ (c) \quad & t = 0 : u = 0. \end{aligned} \quad (22)$$

Proceeding likewise, we also discuss the fractional extensions of the incomplete lumped formulation (the linear case) as well as the lumped formulation of the heat transfer in oil strata [4].

More precisely, we extend the linear case of the incomplete lumped formulation (1), (4), (5) to the equation (20) subject to the boundary condition

$$z = 0 : D_*^{2\beta} u = \frac{\partial^2 u}{\partial x^2} - 2\gamma \frac{\partial u}{\partial x} + \alpha \frac{\partial u}{\partial z}, \quad 0 < x, t < \infty. \quad (23)$$

Similarly, the lumped formulation (1), (4), (5b), (5c) and (6) is extended to the problem of solving the fractional diffusion equation (20) subject to the boundary condition (23) and the conditions (5), where (5a) is replaced by (6).

We extend the problem (7), (8), (9) and (10) to the fractional Lauwerier formulation [5] of the temperature field problem in oil strata that reads as

$$D_*^{2\beta} \Theta = \frac{\partial^2 \Theta}{dz^2}, \quad 0 < x, z, t < \infty, \quad 0 < \beta \leq \frac{1}{2}, \quad (24)$$

subject to the boundary conditions

$$z = 0 : D_*^{2\beta} u = -\gamma \frac{\partial u}{\partial x} - \alpha(u - \Theta), \quad 0 < x, t < \infty, \quad (25)$$

$$z = 0 : D_*^{2\beta} \Theta = \mu \frac{\partial \Theta}{\partial z} + k(u - \Theta), \quad 0 < x, t < \infty, \quad (26)$$

and the conditions

$$\begin{aligned} & \text{(a) } x = 0, z = 0 : u = 1 \\ & \text{(b) } u, \Theta \rightarrow 0 \text{ as } x^2 + z^2 \rightarrow \infty, \\ & \text{(c) } t = 0 : u = \Theta = 0. \end{aligned} \tag{27}$$

3. Auxiliary results

A key role in obtaining the solution (12) is given to the following generalized multiplication theorem proved by A.M. Efros (cf. [1], p.12; [10], pp. 35-36).

LEMMA 1. (Efros' Theorem) *Let be given analytic functions $G(p)$ and $q(p)$ and the relations*

$$F(p) = L[f(t)], \quad e^{-q(p)\tau} G(p) = L[g(t, \tau)]. \tag{28}$$

Then

$$F(q(p))G(p) = L \left[\int_0^\infty f(\tau)g(t, \tau)d\tau \right]. \tag{29}$$

P r o o f. Provided that the order of integration can be reversed and using the second relation of (28), the right-hand side of (29) is given by

$$\begin{aligned} \int_0^\infty e^{-pt} \int_0^\infty f(\tau)g(t, \tau)d\tau dt &= \int_0^\infty f(\tau) \int_0^\infty g(t, \tau)e^{-pt} dt d\tau \\ &= \int_0^\infty f(\tau)e^{-q(p)\tau} d\tau G(p) \end{aligned}$$

and hence, by the left hand-side of (29) the statement follows. ■

In the particular case $q(p) = p$ it holds

$$L[g(t, \tau)] = e^{-p\tau} G(p)$$

and by the *p-shift theorem*, $g(t, \tau) = g(t - \tau)$. Hence, formula (29) becomes

$$F(p)G(p) = L \left[\int_0^t f(\tau)g(t - \tau)d\tau \right],$$

since for the original functions it holds $g(t - \tau) = 0$ for $\tau > t$. The last formula shows that Efros' theorem is a generalization of a convolution theorem for the Laplace transform.

In studying the time fractional diffusion equation (19), the fundamental solution of the basic Cauchy problem can be expressed by an auxiliary function defined as [17],

$$M(z; \beta) = \frac{1}{2\pi i} \int_{H_a} \frac{1}{\sigma^{1-\beta}} e^{\sigma - z\sigma^\beta} d\sigma, \quad 0 < \beta < 1,$$

where H_a denotes the Hankel path of integration that begins at $\sigma = -\infty - ib_1$ ($b_1 > 0$), encircles the branch cut that lies along the negative real axis, and ends up at $\sigma = -\infty + ib_2$ ($b_2 > 0$). It is also proved that

$$M(z; \beta) = W(-z; -\beta, 1 - \beta),$$

where

$$W(z; \lambda, \mu) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)} = \frac{1}{2\pi i} \int_{H_a} \sigma^\mu e^{\sigma + z\sigma^{-\lambda}} d\sigma, \quad \lambda > -1, \mu > 0,$$

is an entire function of z referred to as the Wright's function (cf. [11], vol. III, Ch. 18). In the particular case $\beta = \frac{1}{2}$ it holds

$$M(z; \frac{1}{2}) = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{2}\right)^m \frac{z^{2m}}{(2m)!} = \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}}. \quad (30)$$

Further, the function

$$N(\xi; \beta) = \frac{1}{2\pi i} \int_{H_a} \frac{1}{\sigma} e^{\sigma - \xi \sigma^{2\beta}} d\sigma,$$

will be useful. By adopting the approach in [17], we obtain the following auxiliary result.

LEMMA 2. *If $0 < t, \tau, z < \infty$,*

$$g_1(t, \tau, z; \beta) = \frac{(b\tau + \frac{z}{a})\beta}{t^{\beta+1}} M\left(\frac{b\tau + \frac{z}{a}}{t^\beta}; \beta\right) \quad (31)$$

and

$$g_2(t, \tau; \beta) = N\left(\frac{\tau}{t^{2\beta}}; \beta\right), \quad (32)$$

where $0 < \beta \leq \frac{1}{2}$, then the following equalities hold

$$\begin{aligned} \text{(i)} \quad e^{-(\frac{z}{a} + b\tau)p^\beta} &= L[g_1(t, \tau, z; \beta)], \\ \text{(ii)} \quad \frac{1}{p} e^{-\tau p^{2\beta}} &= L[g_2(t, \tau; \beta)]. \end{aligned}$$

P r o o f. Part (i) is a direct consequence of [17].

To prove part (ii), consider the Laplace transform

$$L[g_2(\tau, t; \beta)] = \frac{1}{p} e^{-\tau p^{2\beta}}.$$

According to the inversion formula for the Laplace transform,

$$g_2(\tau, t; \beta) = \frac{1}{2\pi i} \int_{H_a} \frac{1}{p} e^{pt - \tau p^{2\beta}} dp.$$

Putting $\sigma = pt$ and introducing the variable $\eta = \frac{\tau}{t^{2\beta}}$, it follows

$$g_2(\tau, t; \beta) = N(\eta; \beta) = \frac{1}{2\pi i} \int_{H_a} \frac{1}{\sigma} e^{\sigma - \eta \sigma^{2\beta}} d\sigma,$$

where $N(\eta; \beta)$ is the auxiliary function. Using Taylor's representation of the exponential function and Hankel's representation of the reciprocal Euler gamma function, we arrive to the following representation

$$N(\eta; \beta) = \sum_{n=0}^{\infty} \frac{(-1)^n \eta^n}{n! \Gamma(-2\beta n + 1)}.$$

Hence, for $0 < \beta < \frac{1}{2}$ the auxiliary function is Wright function, i.e.

$$N(\eta; \beta) = W(-\eta; -2\beta, 1).$$

In the particular case $\beta = \frac{1}{2}$ it is

$$N(\eta; \frac{1}{2}) = \frac{1}{2\pi i} \int_{H_a} \frac{1}{p} e^{pt} (e^{-\tau p}) dp = H(t - \tau) \quad (33)$$

where $H(t - \tau)$ is the Heaviside function. Thus for $0 < \beta \leq \frac{1}{2}$ we can state that

$$\frac{1}{p} e^{-\tau p^{2\beta}} = L[g_2(t, \tau; \beta)],$$

where

$$g_2(t, \tau; \beta) = N\left(\frac{\tau}{t^{2\beta}}; \beta\right)$$

and that completes the proof of part (ii). ■

Similarly to the above statement, for the function

$$N(\xi; \beta; \lambda) = \frac{1}{2\pi i} \int_{H_a} e^{\sigma - \xi \sigma^{2\beta}} \frac{d\sigma}{\sigma^\lambda}, \lambda > 0, 0 < \beta < \frac{1}{2}, \quad (34)$$

we obtain the following auxiliary result.

LEMMA 3. If $0 < \beta \leq \frac{1}{2}$ and $0 < t, \tau, z < \infty$, then the following relations hold

$$\begin{aligned} \text{(i)} \quad e^{-(z+\mu\tau)p^\beta} &= L[\hat{g}_1(t, \tau, z; \beta)], \quad \mu > 0; \\ \text{(ii)} \quad \frac{1}{p^\lambda} e^{-(\frac{x}{\gamma}+\tau)p^{2\beta}} &= L[\hat{g}_2(t, \tau; \beta; \lambda)], \quad \lambda > 0, x > 0, \end{aligned}$$

where

$$\hat{g}_1(t, \tau, z; \beta) = \frac{(z + \mu\tau)\beta}{t^{\beta+1}} M\left(\frac{z + \mu\tau}{t^\beta}; \beta\right) \quad (35)$$

and

$$\hat{g}_2(t, \tau; \beta; \lambda) = t^{\lambda-1} N\left(\frac{\frac{x}{\gamma} + \tau}{t^{2\beta}}; \beta; \lambda\right). \quad (36)$$

P r o o f. The validity of (i) is a direct consequence of [16], formulas (3.4) and (3.5).

To prove part (ii) let us consider the Laplace transform

$$L[\hat{g}_2(t, \tau; \beta; \lambda)] = \frac{1}{p^\lambda} e^{-(\frac{x}{\gamma}+\tau)p^{2\beta}}, \quad \lambda > 0.$$

According to the inversion formula for the Laplace transform,

$$\hat{g}_2(t, \tau; \beta; \lambda) = \frac{1}{2\pi i} \int_{H_a} e^{pt} \left[\frac{1}{p^\lambda} e^{-(\frac{x}{\gamma}+\tau)p^{2\beta}} \right] dp.$$

Setting $\sigma = pt$ and $\xi = \frac{\frac{x}{\gamma}+\tau}{t^{2\beta}}$, we obtain

$$\hat{g}_2(t, \tau; \beta; \lambda) = \frac{t^{\lambda-1}}{2\pi i} \int_{H_a} e^{\sigma-\xi\sigma^{2\beta}} \frac{d\sigma}{\sigma^\lambda}.$$

Hence if $0 < 2\beta < 1$,

$$\hat{g}_2(t, \tau; \beta; \lambda) = t^{\lambda-1} W(-\xi; -2\beta; \lambda) = t^{\lambda-1} N(\xi; \beta; \lambda).$$

In particular,

$$\hat{g}_2(t, \tau; \frac{1}{2}; \lambda) = \frac{t^{\lambda-1}}{2\pi i} \int_{H_a} e^{\sigma-\xi\sigma} \frac{d\sigma}{\sigma^\lambda} = \frac{1}{2\pi i} \int_{H_a} e^{pt} \left[\frac{1}{p^\lambda} e^{-(\frac{x}{\gamma}+\tau)p} \right] dp.$$

Referring to [9], formulas (3) and (27) we conclude that

$$\hat{g}_2(t, \tau; \frac{1}{2}; \lambda) = \int_0^t \frac{s^{\lambda-1}}{\Gamma(\lambda)} \delta\left(t - \frac{x}{\gamma} - \tau - s\right) ds,$$

where $\delta(t)$ is the Dirac delta function. It is also worth to mention that if $\lambda = 1$,

$$\hat{g}_2(t, \tau; \frac{1}{2}; 1) = H\left(t - \frac{x}{\gamma} - \tau\right). \quad (37)$$

4. Fractional incomplete lumped formulation (the radial case)

In this section the Laplace transform is used to solve the fractional problem (20), (21) and (22). By using essentially rule (18), we prove the following theorem.

THEOREM 1. *The solution of the radial case of the fractional incomplete formulation (20), (21) and (22) is given by the integral*

$$u(r, z, t) = \frac{2}{\Gamma(\nu)} \int_0^\infty \frac{1}{\tau} \left(\frac{r^2}{4\tau} \right)^\nu e^{-r^2/4\tau} g(t, \tau, z; \beta) d\tau, \quad (38)$$

where

$$g(t, \tau, z; \beta) = g_1(t, \tau, z; \beta) * g_2(t, \tau; \beta) \quad (39)$$

and $g_1(t, \tau, z; \beta)$ and $g_2(t, \tau; \beta)$ are defined by (31) and (32), respectively.

P r o o f. Let $\bar{u}(r, z, p) = L[u(r, z, t)]$. Taking the Laplace transform of (20) and (21) and using the initial condition (22c), we obtain

$$p^{2\beta} \bar{u} = a^2 \frac{\partial^2 \bar{u}}{\partial z^2}, \quad 0 < r, z < \infty, \quad (40)$$

$$z = 0 : p^{2\beta} \bar{u} = \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1 - 2\nu}{r} \frac{\partial \bar{u}}{\partial r} + \alpha \frac{\partial \bar{u}}{\partial z}, \quad 0 < r < \infty, \quad (41)$$

$$\begin{aligned} (a) \quad & r = 0, z = 0 : \bar{u} = \frac{1}{p}, \quad \text{and} \\ (b) \quad & \text{if } r^2 + z^2 \rightarrow \infty, \quad \text{then } \bar{u} \rightarrow 0. \end{aligned} \quad (42)$$

The solution of (40) which remains bounded as $z \rightarrow \infty$ is

$$\bar{u}(r, z, p) = c(r, p) e^{-\frac{p^\beta}{a} z}, \quad (43)$$

where the function $c(r, p)$ has to be determined. Substituting (43) into (41) leads to the ordinary differential equation

$$\frac{d^2 c}{dr^2} + \frac{1 - 2\nu}{r} \frac{dc}{dr} - [p^{2\beta} + bp^\beta] c = 0. \quad (44)$$

It follows from (42) and (43) that

$$c(0, p) = \frac{1}{p}, \quad \lim_{r \rightarrow \infty} c(r, p) = 0. \quad (45)$$

Let us abbreviate $\mu = \sqrt{p^{2\beta} + bp^\beta}$. Then the solution of (44), which remains bounded as $r \rightarrow \infty$ ([14], p. 106, formula (5.4.11)) is given by

$$c(r, p) = c_1(p) \left(\frac{\mu r}{2} \right)^\nu K_\nu(\mu r), \quad (46)$$

where $K_\nu(z)$ is the modified Bessel function of the second kind of order ν and $c_1(p)$ is a constant that must conform with (45). To apply conditions (45), we consider $\lim_{r \rightarrow \infty} c(r, p)$ by referring to the following formula ([26], p.111):

$$K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{2}{z}\right)^\nu \text{ as } z \rightarrow 0. \quad (47)$$

Then it follows from (45) and (46) that

$$c_1(p) = \frac{2}{p\Gamma(\nu)}. \quad (48)$$

Substituting (48) and (46) into (43) gives the Laplace transform of the solution

$$\bar{u}(r, z, p) = \frac{2}{p\Gamma(\nu)} \left(\frac{\mu r}{2}\right)^\nu K_\nu(\mu r) e^{-\frac{p^\beta}{a} z}. \quad (49)$$

To make use of Lemma 1, we represent (49) in the form

$$\bar{u}(r, z, p) = G(p; \beta) F[q(p; \beta)], \quad (50)$$

where

$$G(p; \beta) = \frac{2}{p\Gamma(\nu)} e^{-\frac{p^\beta}{a} z} \text{ and } q(p; \beta) = p^{2\beta} + bp^\beta. \quad (51)$$

It is well-known ([9], p.345, formula 2.10.128) that

$$p^{\nu/2} K_\nu(\alpha\sqrt{p}) = L \left[\frac{\alpha^\nu}{(2t)^{\nu+1}} e^{-\frac{\alpha^2}{4t}} \right].$$

Then from (49) and (50) it follows

$$F(p) = \left(\frac{r\sqrt{p}}{2}\right)^\nu K_\nu(r\sqrt{p}) = L[f(t; \beta)],$$

where

$$f(t; \beta) = \left(\frac{r}{2}\right)^\nu \frac{r^\nu}{(2t)^{\nu+1}} e^{-\frac{r^2}{4t}}, \quad (52)$$

and furthermore,

$$G(p; \beta) e^{-\tau q(p; \beta)} = \frac{2}{\Gamma(\nu)} \cdot \frac{1}{p} e^{-\tau p^{2\beta}} e^{-(b\tau + \frac{\tau}{a} p^\beta)}.$$

Then Lemma 2 and the convolution theorem yield

$$G(p; \beta) e^{-\tau q(p; \beta)} = \frac{2}{\Gamma(\nu)} L[g(t, \tau, z; \beta)], \quad (53)$$

where $g(t, \tau, z; \beta)$, $g_1(t, \tau, z; \beta)$ and $g_2(t, \tau; \beta)$ are defined by (39), (31) and (32), respectively. Taking into account (52) and (53), by Lemma 1 we obtain the solution of the radial case of the fractional incomplete lumped formulation in the form (38). ■

COROLLARY 1. *For the particular case $\beta = \frac{1}{2}$ the result (38) in Theorem 1 yields the representation (12).*

P r o o f. To make sure that the solution (12) occurs as a particular case of the solution obtained, consider the convolution (39)

$$\begin{aligned} g_1(t, \tau, z; \beta) * g_2(t, \tau; \beta) &= \int_0^t g_1(t-s, \tau, z; \beta) g_2(s, \tau; \beta) ds \\ &= \int_0^t \frac{(b\tau + z/a)\beta}{(t-s)^{\beta+1}} M\left(\frac{b\tau + \frac{z}{a}}{(t-s)^\beta}; \beta\right) N\left(\frac{\tau}{s^{2\beta}}; \beta\right) ds \end{aligned}$$

for the case $\beta = \frac{1}{2}$. In accordance with (30) and (33) the convolution becomes

$$\begin{aligned} g_1(t, \tau, z; \frac{1}{2}) * g_2(t, \tau, z; \frac{1}{2}) &= H(t-\tau) \frac{1}{2\sqrt{\pi}} \int_\tau^t \frac{b\tau + \frac{z}{a}}{(t-s)^{3/2}} e^{-\left(\frac{b\tau + \frac{z}{a}}{2\sqrt{t-s}}\right)^2} ds \\ &= \frac{H(t-\tau)}{2\sqrt{\pi}} \int_0^{t-\tau} \frac{b\tau + \frac{z}{a}}{(t-\tau-\xi)^{3/2}} e^{-\left(\frac{b\tau + \frac{z}{a}}{2\sqrt{t-\tau-\xi}}\right)^2} d\xi \\ &= \frac{2}{\sqrt{\pi}} H(t-\tau) \int_{\frac{b\tau + \frac{z}{a}}{2\sqrt{t-\tau}}}^\infty e^{-w^2} dw = H(t-\tau) \operatorname{erfc}\left(\frac{b\tau + \frac{z}{a}}{2\sqrt{t-\tau}}\right). \end{aligned}$$

Hence, if $\beta = \frac{1}{2}$ the solution (38) yields the formulae (12). ■

5. Fractional incomplete lumped formulation (the linear case)

In this section we focus on the fractional incomplete lumped formulation (linear case) (5), (20), (23). Following the same approach as in the case of radial case, we prove the following theorem.

THEOREM 2. *The solution of the fractional incomplete lumped formulation (the linear case) (5), (20) and (23) is given by the integral*

$$u(x, z, t) = \frac{x e^{\gamma x}}{2\sqrt{\pi}} \int_0^\infty \tau^{-3/2} e^{-\frac{x^2}{4\tau} - \tau\gamma^2} g(t, \tau, z; \beta) d\tau, \quad (54)$$

where $g(t, \tau, z; \beta)$ is defined by (39) in terms of the functions $g_1(t, \tau, z; \beta)$ and $g_2(t, \tau; \beta)$ given by (31) and (32) respectively.

P r o o f. Let $\bar{u}(x, z, p) = L[u(x, z, t)]$. According to (5b) and (18) the Laplace transform reduces (5), (20) and (23) to

$$p^{2\beta} \bar{u}(x, z, p) = a^2 \frac{\partial^2 \bar{u}(x, z, p)}{\partial z^2}, \quad 0 < x, z < \infty, \quad (55)$$

$$z = 0 : p^{2\beta} \bar{u}(x, 0, p) = \frac{\partial^2 \bar{u}(x, 0, p)}{\partial x^2} - 2\gamma \frac{\partial \bar{u}(x, 0, p)}{\partial x} + \alpha \frac{\partial \bar{u}(x, 0, p)}{\partial z}, \quad (56)$$

$$\begin{aligned} \text{(a)} \quad & z = x = 0 : \bar{u}(0, 0, p) = \frac{1}{p}, \quad \text{and} \\ \text{(b)} \quad & \text{if } x^2 + z^2 \rightarrow \infty \text{ then } \bar{u}(x, z, p) \rightarrow 0. \end{aligned} \quad (57)$$

By using (57) the solution of (55), which remains bounded as $z \rightarrow \infty$, will be

$$\bar{u}(x, z, p) = c(x, p) e^{\frac{-p^\beta}{a} z}, \quad (58)$$

where the function $c(x, p)$ has to be determined. The substitution of (58) into (56) leads to the differential equation

$$\frac{d^2}{dx^2} c(x, p) - 2\gamma \frac{d}{dx} c(x, p) - (p^{2\beta} + bp^\beta) c(x, p) = 0. \quad (59)$$

From (57) and (58) it is clear that the solution of (59) must satisfy the conditions

$$c(0, p) = \frac{1}{p}, c(x, p) \rightarrow 0 \text{ as } x \rightarrow \infty. \quad (60)$$

Solving the equation (59) under the conditions (60) it follows

$$c(x, p) = \frac{1}{p} e^{(\gamma - \sqrt{q(p; \beta)})x},$$

where $q(p; \beta) = \gamma^2 + p^{2\beta} + bp^\beta$. Hence, the solution of (55), (56) and (57) is given by

$$\bar{u}(x, z, p) = \frac{e^{\gamma x}}{p} e^{-\frac{p^\beta}{a} z - x \sqrt{q(p; \beta)}}. \quad (61)$$

Then by Lemma 1 it follows that

$$\bar{u}(x, z, p) = G(p; \beta) F[q(p; \beta)], \quad (62)$$

where

$$G(p; \beta) = \frac{e^{\gamma x}}{p} e^{-\frac{p^\beta}{a} z} \quad \text{and} \quad F[q(p; \beta)] = e^{-x \sqrt{q(p; \beta)}}.$$

It is well-known ([28], Appendix B, formula (85)) that

$$F(p) = e^{-x\sqrt{p}} = L \left[\frac{x e^{-x^2/4t}}{2t\sqrt{\pi t}} \right] = L[f(t)] \quad (63)$$

and by Lemma 1 it follows $f(t)$ is deduced. On the other hand, it is clear that

$$e^{-\tau q(p;\beta)} G(p; \beta) = e^{\gamma x - \tau \gamma^2} e^{-p^{2\beta} \tau} \frac{1}{p} e^{-(\frac{z}{a} + b\tau)p^\beta}.$$

The latest equation allows to apply Lemma 2 and the convolution theorem for the Laplace transform that yields

$$e^{-\tau q(p;\beta)} G(p; \beta) = e^{\gamma x - \tau \gamma^2} L[g(t, \tau, z; \beta)], \quad (64)$$

where $g(t, \tau, z; \beta)$ is defined by (39). Taking into account (63) and (64), by Lemma 1 we obtain the solution of the fractional lumped formulation (linear case) in the desired form (54). ■

COROLLARY 2. *For the particular case $\beta = \frac{1}{2}$, the result (54) of Theorem 2 yields the solution (13).*

The proof of Corollary 1 judges the validity of Corollary 2. It is also obvious that the solution $u_s(x, t)$ (see (13)) follows from (54) in case $z = 0$.

6. Fractional lumped formulation

Consider now the fractional lumped formulation as we introduced the problem of solving the fractional heat equation (20) subject to the boundary condition (23) and the conditions (5), where (5) (a) is replaced by (6). We apply the same approach as in the preceding sections to prove the following statement.

THEOREM 3. *The solution of the fractional lumped formulation is given by the integral*

$$u(x, z, t) = 2\gamma e^{\gamma x} \int_0^\infty \left[\frac{1}{\sqrt{\pi\tau}} e^{-\left(\frac{x^2}{4\tau} + \tau\gamma^2\right)} - \gamma e^{\gamma x} \operatorname{erfc}\left(\frac{x}{2\sqrt{\tau}} + \gamma\sqrt{\tau}\right) \right] g(t, \tau, z; \beta) d\tau, \quad (65)$$

where $g(t, \tau, z; \beta)$ is given by (39) in terms of the functions $g_1(t, \tau, z; \beta)$ and $g_2(t, \tau; \beta)$ defined by (31) and (32), respectively.

P r o o f. Let $\bar{u}(x, z, p) = L[(u(x, z, t))]$. Applying the Laplace transform, the fractional lumped formulation reduces to the following boundary value problem

$$p^{2\beta} \bar{u}(x, z, p) = a^2 \frac{\partial^2 \bar{u}(x, z, p)}{\partial z^2}, \quad 0 < x, z < \infty, \quad (66)$$

$$z = 0 : p^{2\beta} \bar{u}(x, 0, p) = \frac{\partial^2 u(x, 0, p)}{\partial x^2} - 2\gamma \frac{\partial u(x, 0, p)}{\partial x} + \alpha \frac{\partial u(x, 0, p)}{\partial z}, \quad (67)$$

$$\begin{aligned} \text{(a)} \quad z = x = 0 : \frac{\partial \bar{u}(0, 0, p)}{\partial x} - 2\gamma \bar{u}(0, 0, p) &= -\frac{2\gamma}{p}, \quad \text{and} \\ \text{(b)} \quad \text{if } x^2 + z^2 \rightarrow \infty \quad \text{then} \quad \bar{u}(x, z, p) &\rightarrow 0. \end{aligned} \quad (68)$$

Proceeding likewise as in the previous two sections, we find that the solution of (66) that satisfies both (67) and (68b) is of the form

$$\bar{u}(x, z, p) = c(p) e^{(\gamma - \sqrt{q(p; \beta)})x} e^{-\frac{p^\beta}{a}z}, \quad (69)$$

where $q(p; \beta) = \gamma^2 + p^{2\beta} + bp^\beta$. Substituting (69) into (68a) leads to

$$c(p) = \frac{2\gamma}{p(\gamma + \sqrt{q(p; \beta)})},$$

and hence, for the solution of (66), (67) and (68) we obtain

$$\bar{u}(x, z, p) = \frac{2\gamma e^{(\gamma - \sqrt{q(p; \beta)})x}}{p(\gamma + \sqrt{q(p; \beta)})} e^{-\frac{p^\beta}{a}z}. \quad (70)$$

To use Lemma 1, we rewrite (70) as

$$\bar{u}(x, z, p) = G(p; \beta) F(q(p; \beta)), \quad (71)$$

where

$$G(p; \beta) = 2\frac{\gamma}{p} e^{\gamma x} e^{-\frac{p^\beta}{a}z} \quad \text{and} \quad F(q(p; \beta)) = \frac{e^{-x\sqrt{q(p; \beta)}}}{x + \sqrt{q(p; \beta)}}.$$

It is well-known ([28], Appendix B, formula (88)), that

$$F(p) = L \left[\frac{1}{\sqrt{\pi t}} e^{-x^2/4t} - \gamma e^{\gamma x + \gamma^2 t} \operatorname{erfc} \left(\frac{x}{2\sqrt{t}} + \gamma\sqrt{t} \right) \right] = L[f(t)], \quad (72)$$

and by Lemma 1, the function $f(t)$ is deduced. Similarly to the two previous theorems, we represent $G(p; \beta)$ as follows

$$e^{-\tau q(p; \beta)} G(p; \beta) = 2\gamma e^{\gamma x - \tau \gamma^2} e^{-\tau p^{2\beta}} \frac{1}{p} e^{-(\frac{z}{a} + b\tau)p^\beta}.$$

By application of Lemma 2 and the convolution theorem for the Laplace transform, we arrive to the representation

$$e^{-\tau q(p; \beta)} G(p; \beta) = L[2\gamma e^{\gamma x - \tau \gamma^2} g(t, \tau, z; \beta)], \quad (73)$$

where $g(t, \tau, z; \beta)$ is defined by (39). Finally from (72), (73) and Lemma 1 we obtain the desired solution in the form (65). \blacksquare

Similarly to Corollary 1 it is possible to prove the following statement.

COROLLARY 3. As $\beta = \frac{1}{2}$, the result (65) of Theorem 3 yields the solution (14). The temperature of the strata $u_s(x, t)$ (see (14)) is a particular case of (65) when $z = 0$.

7. Fractional Lauwerier formulation

THEOREM 4. If $0 < \beta \leq \frac{1}{2}$ and $\lambda > 0$, the solutions of the fractional Lauwerier formulation of the temperature field problem in oil strata (24), (25), (26) and (27) are given by the integrals

$$u(x, t) = \int_0^\infty e^{-\left(\frac{\alpha x}{\gamma} + \tau k\right)} [\varphi(t, \tau; \beta) * \hat{g}_1(t, \tau, 0; \beta)] I_0 \left(2\sqrt{\frac{\alpha k x}{\gamma}} \tau \right) d\tau, \quad (74)$$

$$\Theta(x, z, t) = k \int_0^\infty e^{-\left(\frac{\alpha x}{\gamma} + \tau k\right)} [\hat{g}_1(t, \tau, z; \beta) * \hat{g}_2(t, \tau, \beta; 1)] I_0 \left(2\sqrt{\frac{\alpha k x}{\gamma}} \tau \right) d\tau, \quad (75)$$

where $\hat{g}_1(t, \tau, z; \beta)$ and $\hat{g}_2(t, \tau, \beta; 1)$ are defined by (35) and (36), respectively, and

$$\varphi(t, \tau; \beta) = \hat{g}_2(t, \tau; \beta; 1 - 2\beta) + \mu \hat{g}_2(t, \tau; \beta; 1 - \beta) + k \hat{g}_2(t, \tau; \beta; 1).$$

P r o o f. Let us denote

$$\bar{u}(x, p) = L[u(x, t)], \bar{\Theta}(x, z, p) = L[\Theta(x, z, t)],$$

where L is the Laplace transformation operator. Applying the Laplace transform to (24), (25), (26) and (27), we obtain in accordance with (18) and (27c),

$$p^{2\beta} \bar{\Theta}(x, z, p) = \frac{\partial^2 \bar{\Theta}(x, z, p)}{\partial z^2}, \quad 0 < x, z < \infty, \quad (76)$$

$$z = 0 : p^{2\beta} \bar{u}(x, p) = -\gamma \frac{\partial \bar{u}(x, p)}{\partial x} - \alpha [\bar{u}(x, p) - \bar{\Theta}(x, z, p)], \quad 0 < x < \infty, \quad (77)$$

$$z = 0 : p^{2\beta} \bar{u}(x, z, p) = \mu \frac{\partial \bar{\Theta}(x, z, p)}{\partial x} + k [\bar{u}(x, p) - \bar{\Theta}(x, z, p)], \quad 0 < x < \infty, \quad (78)$$

$$\begin{aligned} (a) \quad & x = 0 : \bar{u}(0, p) = \frac{1}{p}, \\ (b) \quad & \bar{u}, \bar{\Theta} \rightarrow 0 \quad \text{as} \quad x^2 + z^2 \rightarrow \infty. \end{aligned} \quad (79)$$

The solution of (76) which remains bounded as $z \rightarrow \infty$ is

$$\bar{\Theta}(x, z, p) = c(x, p)e^{-zp^\beta}, \quad (80)$$

where $c(x, p)$ is still unknown function. Substituting (80) into (77) and (78) we have

$$p^{2\beta}\bar{u}(x, p) = -\gamma \frac{\partial \bar{u}(x, p)}{\partial x} - \alpha[\bar{u}(x, p) - c(x, p)], \quad (81)$$

$$p^{2\beta}c(x, p) = -\mu p^\beta c(x, p) + k[\bar{u}(x, p) - c(x, p)]. \quad (82)$$

Solving (82) for $c(x, p)$ we get

$$c(x, p) = \frac{k\bar{u}(x, p)}{p^{2\beta} + \mu p^\beta + k}.$$

The substitution of this representation of $c(x, p)$ into (81) leads to the equation

$$\frac{\partial \bar{u}(x, p)}{\partial x} = -\frac{1}{\gamma} \left(p^{2\beta} + \alpha - \frac{\alpha k}{p^{2\beta} + \mu p^\beta + k} \right) \bar{u}(x, p).$$

The solution of this equation that conform (79a) is

$$\bar{u}(x, p) = \frac{1}{p} \exp \left[\frac{\alpha k x}{\gamma} \frac{1}{p^{2\beta} + \mu p^\beta + k} - \frac{(p^{2\beta} + \alpha)x}{\gamma} \right]. \quad (83)$$

Then obviously,

$$c(x, p) = \frac{k}{p^{2\beta} + \mu p^\beta + k} \frac{1}{p} \exp \left[\frac{\alpha k x}{\gamma} \frac{1}{p^{2\beta} + \mu p^\beta + k} - \frac{(p^{2\beta} + \alpha)x}{\gamma} \right],$$

and from (80) it follows that

$$\bar{\Theta}(x, z, p) = \frac{k}{p^{2\beta} + \mu p^\beta + k} \frac{e^{-zp^\beta}}{p} \exp \left[\frac{\alpha k x}{\gamma} \frac{1}{p^{2\beta} + \mu p^\beta + k} - \frac{(p^{2\beta} + \alpha)x}{\gamma} \right]. \quad (84)$$

To apply Lemma 1 we represent

$$\bar{u}(x, p) = F[q(p; \beta)]G(x, p; \beta), \quad (85)$$

where

$$F[q(p; \beta)] = \frac{1}{q(p; \beta)} \exp \left[\frac{\alpha k x}{\gamma} \frac{1}{q(p; \beta)} \right], \quad (86)$$

and

$$G(x, p; \beta) = \frac{p^{2\beta} + \mu p^\beta + k}{p} \exp \left[-\frac{x}{\gamma} (p^{2\beta} + \alpha) \right]. \quad (87)$$

The well-known formula ([9], p.247, (2.4.82)) enables us to get

$$F(p) = L[f(t)] = L \left[I_0 \left(2 \sqrt{\frac{\alpha k x}{\gamma}} t \right) \right].$$

Further we consider the function

$$e^{-\tau q(p; \beta)} G(x, p; \beta) = \frac{p^{2\beta} + \mu p^\beta + k}{p} \exp \left[- \left(\frac{x}{\gamma} + \tau \right) p^{2\beta} - \tau \mu p^\beta - \left(\frac{\alpha x}{\gamma} + \tau k \right) \right]. \quad (88)$$

From Lemma 3 and the convolution theorem, we obtain consequently,

$$\frac{1}{p^{1-2\beta}} e^{-(\frac{x}{\gamma} + \tau) p^{2\beta}} e^{-\tau \mu p^\beta} = L[\hat{g}_2(t, \tau; \beta; 1 - 2\beta) * \hat{g}_1(t, \tau, 0; \beta)], \quad (89)$$

$$\frac{1}{p^{1-\beta}} e^{-(\frac{x}{\gamma} + \tau) p^{2\beta}} e^{-\tau \mu p^\beta} = L[\hat{g}_2(t, \tau; \beta; 1 - \beta) * \hat{g}_1(t, \tau, 0; \beta)], \quad (90)$$

$$\frac{1}{p} e^{-(\frac{x}{\gamma} + \tau) p^{2\beta}} e^{-\tau \mu p^\beta} = L[\hat{g}_2(t, \tau; \beta; 1) * \hat{g}_1(t, \tau, 0; \beta)]. \quad (91)$$

Taking into account (88) as well as the formulas (89), (90) and (91), we can write

$$e^{-\tau q(p; \beta)} G(x, p; \beta) = L[g(t, \tau)],$$

where

$$g(t, \tau) = e^{-(\frac{\alpha x}{\gamma} + \tau k)} [\varphi(t, \tau; \beta) * \hat{g}_1(t, \tau, 0; \beta)].$$

The application of Lemma 1 leads directly to the solution (74).

Following the same idea, we represent $\bar{\Theta}(x, z, p)$ as the product

$$\bar{\Theta}(x, z, p) = F[q(p; \beta)] G_1(x, p; \beta), \quad (92)$$

where $F[q(p; \beta)]$ is defined by (86) and

$$G_1(x, p; \beta) = \frac{k}{p} e^{-z p^\beta} e^{-\frac{x}{\gamma} p^{2\beta}} e^{-\frac{\alpha x}{\gamma}}.$$

Obviously,

$$e^{-\tau q(p; \beta)} G_1(x, p; \beta) = k e^{-(\frac{\alpha x}{\gamma} + k\tau)} e^{-(z + \mu\tau) p^\beta} \frac{1}{p} e^{-(\frac{x}{\gamma} + \tau) p^{2\beta}}.$$

Then as a direct consequence from Lemma 3 and the convolution theorem, we obtain

$$e^{-\tau q(p; \beta)} G_1(x, p; \beta) = k e^{-(\frac{\alpha x}{\gamma} + k\tau)} [\hat{g}_1(t, \tau, z; \beta) * \hat{g}_2(t, \tau; \beta; 1)]. \quad (93)$$

Then (92), (93) and Lemma 1 lead to the solution (75) and this completes the proof of the theorem. \blacksquare

COROLLARY 4. *For the particular case $\beta = \frac{1}{2}$ the solution (74) reduces to (15).*

P r o o f. Because of (30), we evidently can write

$$\hat{g}_1(t, \tau, 0; \frac{1}{2}) = \frac{\tau\mu}{2\sqrt{\pi}t^{3/2}} e^{-(\frac{\tau\mu}{2\sqrt{t}})^2}. \quad (94)$$

According to (36) and [(11), 17.13, (27)],

$$\begin{aligned} \hat{g}_2(t, \tau; \frac{1}{2}; 0) &= \frac{1}{t} N\left(\frac{\frac{x}{\gamma} + \tau}{t}; \frac{1}{2}; 0\right) = \frac{1}{2\pi i} \int_{H_a} e^{pt} \left[e^{-(\frac{x}{\gamma} + \tau)p} \right] dp \\ &= \delta(t - \frac{x}{\gamma} - \tau) H(t - \frac{x}{\gamma} - \tau). \end{aligned}$$

Likewise, from ([12], 17.13, (3)) and the t -shifting theorem for the Laplace transform it follows that,

$$\begin{aligned} \hat{g}_2(t, \tau; \frac{1}{2}; \frac{1}{2}) &= \frac{1}{t^{1/2}} N\left(\frac{\frac{x}{\gamma} + \tau}{t}; \frac{1}{2}; \frac{1}{2}\right) \\ &= \frac{1}{2\pi i} \int_{H_a} e^{pt} \left[\frac{1}{p^{1/2}} e^{-(\frac{x}{\gamma} + \tau)p} \right] dp = \frac{H(t - \frac{x}{\gamma} - \tau)}{\sqrt{\pi} \sqrt{t - \frac{x}{\gamma} - \tau}}. \end{aligned}$$

Taking into account (37) we obtain that

$$\varphi(t, \tau; \frac{1}{2}) = [\delta(t - \frac{x}{\gamma} - \tau) + \mu \frac{1}{\sqrt{\pi} \sqrt{t - \frac{x}{\gamma} - \tau}} + k] H(t - \frac{x}{\gamma} - \tau). \quad (95)$$

Formulas (94) and (95) yield that in the case $\beta = \frac{1}{2}$ the convolution into the integrand of (74) takes the form

$$\begin{aligned} \varphi(t, \tau; \frac{1}{2}) * \hat{g}_1(t, \tau, 0; \beta) &= \frac{\tau\mu H(t - \frac{x}{\gamma} - \tau)}{2\sqrt{\pi}} \int_0^{t - \frac{x}{\gamma} - \tau} \frac{1}{s^{3/2}} e^{-(\frac{\tau\mu}{2\sqrt{s}})^2} \delta(t - \frac{x}{\gamma} - \tau - s) ds \\ &+ \frac{\tau\mu^2 H(t - \frac{x}{\gamma} - \tau)}{2\pi} \int_{\frac{x}{\gamma} + \tau}^t \frac{1}{(t-s)^{3/2}} e^{-(\frac{\tau\mu}{2\sqrt{t-s}})^2} \frac{1}{\sqrt{s - \frac{x}{\gamma} - \tau}} ds \\ &+ \frac{k\tau\mu}{2\sqrt{\pi}} H(t - \frac{x}{\gamma} - \tau) \int_{\frac{x}{\gamma} + \tau}^t \frac{1}{(t-s)^{3/2}} e^{-(\frac{\tau\mu}{2\sqrt{t-s}})^2} ds. \end{aligned} \quad (96)$$

The third integral in (96) we compute straightforwardly as

$$\begin{aligned} \frac{\tau\mu}{2} \int_{\frac{x}{\gamma} + \tau}^t (t-s)^{3/2} e^{-(\frac{\tau\mu}{2\sqrt{t-s}})^2} ds &= 2 \int_0^{t - \frac{x}{\gamma} - \tau} e^{-\left(\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau - s'}}\right)^2} \\ &\times d\left(\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau - s'}}\right) = 2 \int_{\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau}}}^{\infty} e^{-w^2} dw = \sqrt{\pi} \operatorname{erfc}\left(\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau}}\right). \end{aligned}$$

By using ([10], 3.471, (3)) one can see that

$$\begin{aligned} & \int_{\frac{x}{\gamma}+\tau}^t (t-s)^{-3/2} (s - \frac{x}{\gamma} - \tau)^{-1/2} e^{-(\frac{t\mu}{2\sqrt{t-s}})^2} ds \\ &= \int_0^{t-\frac{x}{\gamma}-\tau} x^{-3/2} (t - \frac{x}{\gamma} - \tau - x)^{-1/2} e^{-(\frac{\tau\mu}{2\sqrt{x}})^2} dx = \frac{2}{\tau\mu} \frac{\sqrt{\pi}}{\sqrt{t - \frac{x}{\gamma} - \tau}} e^{-(\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau}})^2}. \end{aligned}$$

For the first integral in (96) it is clear that

$$\int_0^{t-\frac{x}{\gamma}-\tau} \frac{1}{s^{3/2}} e^{-(\frac{\tau\mu}{2\sqrt{s}})^2} \delta(t - \frac{x}{\gamma} - \tau - s) ds = (t - \frac{x}{\gamma} - \tau)^{-3/2} e^{-(\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau}})^2}.$$

The substitution of these integrals by their expressions into (96) results to

$$\begin{aligned} & \varphi(t, \tau; \frac{1}{2}) * \hat{g}_1(t; \tau, 0; \beta) \\ &= \left\{ \frac{\mu[2(t - \frac{x}{\gamma}) - \tau]}{2\sqrt{\pi}(t - \frac{x}{\gamma} - \tau)^{3/2}} e^{-(\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau}})^2} + k \operatorname{erfc} \left(\frac{\tau\mu}{2\sqrt{t - \frac{x}{\gamma} - \tau}} \right) \right\} H(t - \frac{x}{\gamma} - \tau), \end{aligned}$$

that proves the statement. \blacksquare

COROLLARY 5. *For the particular case $\beta = \frac{1}{2}$ the solution (75) reduces to (16).*

P r o o f. From (30) and (35) it follows that

$$\hat{g}_1(t, \tau, z; \frac{1}{2}) = \frac{z + \mu\tau}{2t^{3/2}} \cdot \frac{1}{\sqrt{\pi}} e^{-(\frac{z+\mu\tau}{2\sqrt{t}})^2}.$$

According to (37), the convolution into the integral of (75) takes the form

$$\begin{aligned} \hat{g}_1(t, \tau, z; \frac{1}{2}) * \hat{g}_2(t, \tau; \frac{1}{2}; 1) &= \int_0^t \frac{z + \mu\tau}{2\sqrt{\pi}(t-s)^{3/2}} e^{-(\frac{z+\mu\tau}{2\sqrt{t-s}})^2} H(s - \frac{x}{\gamma} - \tau) ds \\ &= H(t - \frac{x}{\gamma} - \tau) \frac{1}{2\sqrt{\pi}} \int_{\frac{x}{\gamma}+\tau}^t \frac{z + \mu\tau}{(t-s)^{3/2}} e^{-(\frac{z+\mu\tau}{2\sqrt{t-s}})^2} ds \\ &= H(t - \frac{x}{\gamma} - \tau) \frac{1}{2\sqrt{\pi}} \int_0^{t-\frac{x}{\gamma}-\tau} \frac{z + \mu\tau}{(t - \frac{x}{\gamma} - \tau - \xi)^{3/2}} e^{-(\frac{z+\mu\tau}{2\sqrt{t - \frac{x}{\gamma} - \tau - \xi}})^2} d\xi \\ &= H(t - \frac{x}{\gamma} - \tau) \frac{2}{\sqrt{\pi}} \int_0^{t-\frac{x}{\gamma}-\tau} e^{-(\frac{z+\mu\tau}{2\sqrt{t - \frac{x}{\gamma} - \tau - \xi}})^2} d \left(\frac{z + \mu\tau}{2\sqrt{t - \frac{x}{\gamma} - \tau - \xi}} \right) \\ &= H(t - \frac{x}{\gamma} - \tau) \frac{2}{\sqrt{\pi}} \int_{\frac{z+\mu\tau}{2\sqrt{t - \frac{x}{\gamma} - \tau}}}^{\infty} e^{-w^2} dw = H(t - \frac{x}{\gamma} - \tau) \operatorname{erfc} \left(\frac{z + \mu\tau}{2\sqrt{t - \frac{x}{\gamma} - \tau}} \right), \end{aligned}$$

and this proves the statement. \blacksquare

References

- [1] M.Ya. Antimirov, A.A. Kolyshkin, R. Vaillancourt, *Applied Integral Transforms*. AMS, Providence, RI - USA (1993).
- [2] N.A. Avdonin, Some formulas for calculating the temperature field of a stratum subject to thermal injection. *Izvestia VUZ, Neft'i Gaz* **7** (1964), 37-41 (Russian).
- [3] Y. Ben Nakhi, S.L. Kalla, Some boundary value problems of temperature fields in oil strata. *Appl. Math. and Comput.* **146** (2003), 105-119.
- [4] L. Boyadjiev, R. Scherer, Fractional extensions of the temperature field problem in oil strata. *Kuwait J. Sci. Engrg.* **31**, No 2 (2004), 15-32.
- [5] L. Boyadjiev, O. Kamenov, S.L.Kalla, On the Lauwerier formulation of the temperature field problem in oil strata. *Int. J. Math. Math. Sci.* **2005**, No 10 (2005), 1577-1588.
- [6] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, II. *Geophysical Journal of the Royal Astronomical Society* **13** (1967), 529-539.
- [7] M. Caputo, F. Mainardi, A new dissipation model based on memory mechanism. *Pure and Applied Geophysics* **91** (1971), 134-147.
- [8] J. Crank, *The Mathematics of Diffusion* (2nd Ed.). Clarendon Press, Oxford (1979).
- [9] V.A. Ditkin, A. P. Prudnikov, *Formulaire pour le calcul operationnel*. Masson et Cie, Paris (1967).
- [10] V.A. Ditkin, A.P. Prudnikov, *Integral Transforms and Operational Calculus*. Pergamon Press, Oxford-N.York (1965).
- [11] A. Erdélyi (Ed.), *Higher Transcendental Functions*. Vol. I, II, III, McGraw-Hill, New York (1955).
- [12] I.S. Gadshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products*. Academic Press, N. York (1980).
- [13] R. Gorenflo, R. Rutman, R. 1995. On ultraslow and on intermediate processes. In: *"Transform Methods and Special Functions, Sofia'94"* (Proc. 1st Workshop, Eds. P. Russev, I. Dimovski and V. Kiryakova), SCT Publishers, Singapore (1995).
- [14] N.N. Lebedev, *Special Functions and Their Applications*. Prentice Hall, Englewood Cliffs, N.J. (1965).
- [15] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation. In: *Waves and Stability in Continuous Media* (Eds. S. Rionero and T. Ruggeri), World Scientific, Singapore (1994), 246-251.

- [16] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos, Solitons and Fractals* **7** (1996), 1461-1477.
- [17] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation. *Applied Mathematics Letters* **9**, No 6 (1996), 23-28.
- [18] B. Mandelbrot, *The Fractal Geometry of Nature*. Freeman, San Francisco (1982).
- [19] R. Metzler, W.G. Glöckle, T.F. Nonnenmacher, Fractional model equation for anomalous diffusion. *Physics A* **211** (1994), 13-24.
- [20] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, N. York (1993).
- [21] R.R. Nigmatullin, To the theoretical explanation of the "universal response". *Phys. Sta. Sol. (b)* **123** (1984), 739-745.
- [22] R.R. Nigmatullin, The realization of the generalized transfer equation in a medium with fractal geometry. *Phys. Sta. Sol. (b)* **133** (1986), 425-430.
- [23] K.B. Oldham, J. Spanier, The replacement of Fick's law by a formulation involving semidifferentiation. *J. Electroanal. Chem. and Interfacial Electrochem.* **26** (1970), 331-341.
- [24] K.B. Oldham, J. Spanier, *The Fractional Calculus*. Academic Press, N. York (1974).
- [25] I. Podlubny, *Fractional Differential Equations (An Introduction to Fractional Derivates, Fractional Differential Equations, to Methods of Their Solution and some of Their Applications)*. Academic Press, San Diego (1999).
- [26] S.G. Samko, A.A. Kilbas, O.I. Marichev, *The Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, Amsterdam (1993).
- [27] W.R. Schneider, W. Wyss, Fractional diffusion and wave equation. *J. Math. Phys.* **30**, No 1 (1989), 134-144.
- [28] M.R. Spiegel, *Theory and Problems of Laplace Transforms*. Schaum Publishing Co., N. York (1965).

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